A conjecture implying the existence of non-convex Chebyshev sets in infinite-dimensional Hilbert spaces

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Abstract. In this paper, we propose the study of a conjecture whose positive solution would provide an example of a non-convex Chebyshev set in an infinite-dimensional real Hilbert space.

Keywords: Chebyshev set, Hilbert space, convexity.

AMS 2000 Subject Classification: 41A50, 41A65.

Here and in the sequel, $(X, \langle \cdot, \cdot \rangle)$ is a separable real Hilbert space, with norm $\| \cdot \|$. A non-empty set $C \subset X$ is said to be a Chebyshev set if, for each $x \in X$, there exists a unique $y \in C$ such that

$$\|x - y\| = \inf_{z \in C} \|x - z\| \ .$$

Clearly, each closed convex set is a Chebyshev one. A natural question is: must any Chebyshev $C \subset X$ be convex ? We refer to the surveys [1], [5] for a thorough discussion of the subject. In particular, it is well-known that any sequentially weakly closed Chebyshev set $C \subset X$ is convex. Hence, if X is finite-dimensional, the answer to the above question is "yes".

However, since [7], it is a quite common feeling that if X is infinite-dimensional, then X contains some non-convex Chebyshev set (see also [6] for a recent contribution in this direction). Maybe, this is the most important conjecture in best approximation theory.

A much more recent (and less known) problem is: if $f : X \to \mathbf{R}$ is a lower semicontinuous function such that, for each $y \in X$ and each $\lambda > 0$, the function $x \to \|x - y\|^2 + \lambda f(x)$ has a unique global minimum, must f be convex ? For this problem too, the answer is "yes" if X is finite-dimensional ([11], Corollary 3.8). See also Corollary 5.2 of [2] for another partial answer.

The aim of the present paper is to show that if the second problem has a qualified negative answer, then the same happens for the first one.

In the sequel, $L^2([0, 1], X)$ is the usual space of all (equivalence classes of) measurable functions $u : [0, 1] \to X$ such that $\int_0^1 \|u(t)\|^2 dt < +\infty$, endowed with the scalar product

$$\langle u, v \rangle_{L^2_X} = \int_0^1 \langle u(t), v(t) \rangle dt$$

The norm induced by $\langle \cdot, \cdot \rangle_{L^2_X}$ is denoted by $\| \cdot \|_{L^2_X}$.

Let us start with the following

DEFINITION 1. - Let Y be a non-empty set and $\mathcal{F}$ a family of subsets of Y.

We say that $\mathcal{F}$ has the compactness-like property if every subfamily of $\mathcal{F}$ satisfying the finite intersection property has a non-empty intersection.

We have the following characterization which is due to C. Costantini ([3]):

PROPOSITION 1. - *Let Y be a non-empty set, let $\mathcal{F}$ be a family of subsets of Y and let τ be the topology on Y generated by the family $\{Y \setminus C\}_{C \in \mathcal{F}}$.*

Then, the following assertions are equivalent:
(i) Each member of $\mathcal{F}$ is τ-compact.
(ii) The family $\mathcal{F}$ has the compactness-like property.
(iii) The space Y is τ-compact.

We then formulate the following

CONJECTURE 1. - If X is infinite-dimensional, there exist a non-convex Borel function $f : X \to \mathbf{R}$, $r \in]\inf_X f, \sup_X f[$ and $\gamma \in]0, +\infty]$, with the following properties:
(a)

$$\sup_{x \in X} \frac{|f(x)|}{1 + \|x\|^2} < +\infty \ ;$$

(b) for each $y \in X$ and each $\lambda \in]0, \gamma[$, the function

$$x \to \|x - y\|^2 + \lambda f(x)$$

has a unique global minimum in X, say $\hat{x}_{y,\lambda}$; moreover, the map $y \to \hat{x}_{y,\lambda}$ is Borel and one has

$$\|\hat{x}_{y,\lambda}\| \leq c_\lambda (1 + \|y\|)$$

where c_λ is independent of y ;
(c) if $\gamma < +\infty$, for each $y \in f^{-1}(]r, +\infty[)$, the function

$$x \to \|x - y\|^2 + \gamma f(x)$$

has no global minima in X ;
(d) for each $v \in L^2([0, 1], X)$, with $\int_0^1 f(v(t))dt > r$, the family

$$\left\{ \left\{ u \in L^2([0,1], X) : \int_0^1 \|u(t) - v(t)\|^2 dt + \lambda \int_0^1 f(u(t))dt \leq \rho \right\} : \lambda \in]0, \gamma[, \rho \in \mathbf{R} \right\}$$

has the compactness-like property.

Our result reads as follows:

THEOREM 1. - *Assume that Conjecture 1 is true and let f be a function satisfying it.*

Then,

$$\left\{u \in L^2([0,1],X) : \int_0^1 f(u(t))dt \leq r\right\}$$

is a non-convex Chebyshev set.

To prove Theorem 1, we need the following two results.

THEOREM A. - *Let Y be a non-empty set, $\eta \in]0,+\infty]$ and $\varphi, \psi : Y \to \mathbf{R}$ two functions such that the function $\varphi + \lambda\psi$ has a unique global minimum if $\lambda \in [0, \eta[$, while has no global minima if $\eta < +\infty$ and $\lambda = \eta$. Moreover, if y_0 is the only global minimum of φ, assume that $\inf_Y \psi < \psi(y_0)$. Finally, assume that the family*

$$\{\{y \in Y : \varphi(y) + \lambda\psi(y) \leq \rho\} : \lambda \in]0, \eta[, \rho \in \mathbf{R}\}$$

has the compactness-like property.

Then, for each $\rho \in]\inf_Y \psi, \psi(y_0)[$, the restriction of the function φ to $\psi^{-1}(\rho)$ has a unique global minimum.

THEOREM B. - *Let $f : X \to \mathbf{R}$ be a Borel function such that*

$$\sup_{x \in X} \frac{|f(x)|}{1 + \|x\|^2} < +\infty \ .$$

Assume that, for some $\rho \in]\inf_X f, \sup_X f[$, the set

$$\left\{u \in L^2([0,1],X) : \int_0^1 f(u(t))dt \leq \rho\right\}$$

is weakly closed.

Then, f is convex.

Theorem A, via Proposition 1, is a direct consequence of a variant of Theorem 1 of [9] (see also the proof of Theorem 1 of [10]), while Theorem B has been proved by R. Landes in [8].

Proof of Theorem 1. Fix $\lambda \in]0, \gamma[$, $v \in L^2([0,1],X)$, with $\int_0^1 f(v(t))dt > r$, and put

$$\omega_{v,\lambda}(t) = \hat{x}_{v(t),\lambda}$$

for all $t \in [0,1]$. From (a) and (b), it clearly follows that the function $\omega_{v,\lambda}$ belongs to $L^2([0,1],X)$. If $u \in L^2([0,1],X)$ and $u \neq \omega_{v,\lambda}$, we have

$$\|\omega_{v,\lambda}(t) - v(t)\|^2 + \lambda f(\omega_{v,\lambda}(t)) \leq \|u(t) - v(t)\|^2 + \lambda f(u(t))$$

for all $t \in [0,1]$, the inequality being strict in a subset of $[0,1]$ with positive measure. Then, by integrating, we get

$$\int_0^1 \|\omega_{v,\lambda}(t) - v(t)\|^2 dt + \int_0^1 \lambda f(\omega_{v,\lambda}(t))dt < \int_0^1 \|u(t) - v(t)\|^2 dt + \lambda \int_0^1 f(u(t))dt \ .$$

Therefore, $\omega_{v,\lambda}$ is the only global minimum in $L^2([0,1],X)$ of the functional

$$u \to \int_0^1 \|u(t)-v(t)\|^2 dt + \lambda \int_0^1 f(u(t))dt \ .$$

Now, assume that $\gamma < +\infty$. Put

$$A_v = \{t \in [0,1] : f(v(t)) > r\} \ .$$

Since $\int_0^1 f(v(t)dt > r$, the measure of A_v is positive. We show that the functional

$$u \to \int_0^1 \|u(t)-v(t)\|^2 dt + \gamma \int_0^1 f(u(t))dt$$

has no global minima in $L^2([0,1],X)$. Indeed, fix $u \in L^2([0,1],X)$. It is easy to check that the function $(t,x) \to \|x-v(t)\|^2 + \gamma f(x)$ is $\mathcal{L}([0,1]) \otimes \mathcal{B}(X)$-measurable, where $\mathcal{L}([0,1])$ and $\mathcal{B}(X)$ denote the Lebesgue and the Borel σ-algebras of subsets of $[0,1]$ and X, respectively. So, by Theorem 2.6.40 of [4], the function $t \to \inf_{x\in X}(\|x-v(t)\|^2 + f(x))$ is measurable. On the other hand, in view of (c), we have

$$\inf_{x\in X}(\|x-v(t)\|^2 + \gamma f(x)) < \|u(t)-v(t)\|^2 + \gamma f(u(t))$$

for all $t \in A_v$. Consequently, we can apply Theorem 4.3.7 of [4] to get a measurable function $\xi : [0,1] \to X$ such that

$$\|\xi(t)-v(t)\|^2 + \gamma f(\xi(t)) < \|u(t)-v(t)\|^2 + \gamma f(u(t))$$

for all $t \in A_v$. Finally, choose a set $B \subset A$ with positive measure such that ξ is bounded in B and put

$$w(t) = \begin{cases} \xi(t) & \text{if } t \in B \\ \\ u(t) & \text{if } t \in [0,1] \setminus B \ . \end{cases}$$

Clearly, $w \in L^2([0,1],X)$ and one has

$$\int_0^1 \|w(t)-v(t)\|^2 dt + \gamma \int_0^1 f(w(t))dt < \int_0^1 \|u(t)-v(t)\|^2 dt + \gamma \int_0^1 f(u(t))dt$$

which proves our claim. At this point, we can apply Theorem A taking

$$Y = L^2([0,1],X) \ ,$$

$$\eta = \gamma \ ,$$

$$\varphi(u) = \|u-v\|^2_{L^2_X}$$

and

$$\psi(u) = \int_0^1 f(u(t))dt \ .$$

Then, there exists a unique $u \in \psi^{-1}(r)$ such that

$$\|v-u\|_{L^2_X} = \mathrm{dist}(v, \psi^{-1}(r)) \ .$$

We now claim that such an u is the unique point of $\psi^{-1}(]-\infty, r])$ such that

$$\|v-u\|_{L^2_X} = \mathrm{dist}(v, \psi^{-1}(]-\infty, r])) \ .$$

This amounts to show that if $w \in \psi^{-1}(]-\infty, r])$ is such that

$$\|v-w\|_{L^2_X} = \mathrm{dist}(v, \psi^{-1}(]-\infty, r])) \ , \tag{1}$$

then $\psi(w) = r$. Arguing by contradiction, assume that $\psi(w) < r$. For each measurable set $A \subset [0,1]$, put

$$h_A(t) = \begin{cases} v(t) & \text{if } t \in A \\ \\ w(t) & \text{if } t \in [0,1] \setminus A \ . \end{cases}$$

Also, set

$$D = \{h_A : A \subset [0,1],\ A \text{ measurable}\} \ .$$

It is not hard to check that D is decomposable ([4], p. 452). Moreover, it is clear that $v, w \in D$ and that

$$\|v-h\| < \|v-w\| \tag{2}$$

for all $h \in D \setminus \{v, w\}$. By Corollary 4.5.13 of [4], the set $\psi(D)$ is an interval. Consequently, there exists $h \in D \setminus \{v, w\}$ such that $\psi(h) = r$. This implies a contradiction, in view of (1) and (2). So, $\psi^{-1}(]-\infty, r])$ is a Chebyshev set in $L^2([0,1], X)$. Finally, this set is not convex. Indeed, if it was convex, being closed, it would be weakly closed. Then, by Theorem B, the function f would be convex, against the assumptions. $\triangle$

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